JACOB'S LADDERS, THE ITERATIONS OF JACOB'S LADDER $\varphi_1^k(t)$ AND ASYMPTOTIC FORMULAE FOR THE INTEGRALS OF THE PRODUCTS $Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$ FOR ARBITRARY FIXED $n\in\mathbb{N}$

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Abstract. In this paper we introduce the iterations $\varphi_1^k(t)$ of the Jacob's ladder. It is proved, for example, that the mean-value of the product

$$Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$$

over the segment $[T,T+U]$ is asymptotically equal to $\ln^{n+1}T$. Nor the case $n=1$ cannot be obtained in known theories of Balasubramanian, Heath-Brown and Ivic.

1. Results

Let

$$y=\frac{1}{2}\varphi(t)=\varphi_1(t);\ \varphi_1^0(t)=t;\ \varphi_1^1(t)=\varphi_1(t),\ \varphi_1^2(t)=\varphi_1(\varphi_1(t)),$$

(1.1)

$$\dots,\ \varphi_1^k(t)=\varphi_1(\varphi_1(\dots(\varphi_1(t))\dots),\ t\in[T,T+U],$$

where $\varphi_1^k(t)$ denotes the kth iteration of the Jacob's ladder $y=\varphi(t),\ t\geq T_0[\varphi_1]$. The following Theorem holds true.

Theorem. Let

$$T\geq T_{00}[\varphi_1,n]=\max\{2T_0[\varphi_1],e^{2(n+1)}\},\ U=T^{1/3+2\epsilon}. \tag{1.2}$$

Then for every fixed $n\in\mathbb{N}$ the following is true

$$\int_T^{T+U}\prod_{k=0}^n Z^2[\varphi_1^k(t)]\mathrm{d}t\sim U\ln^{n+1}T,\ T\to\infty, \tag{1.3}$$

where

$$\varphi_1^k(t)\geq T_0[\varphi_1[,\ k=0,1,\dots,n+1,\ t\in[T,T+U], \tag{A}$$

$$\varphi_1^k(T+U)-\varphi_1^k(t)\sim U,\ k=0,1,\dots,n+1, \tag{B}$$

$$\varphi_1^{k-1}(T)-\varphi_1^k(T+U)\sim(1-c)\frac{T}{\ln T},\ k=0,1,\dots,n, \tag{C}$$

$$\rho\left\{\left[\varphi_1^{k-1}(T),\varphi_1^{k-1}(T+U)\right];\left[\varphi_1^k(T),\varphi_1^k(T+U)\right]\right\}\sim(1-c)\frac{T}{\ln T}, \tag{D}$$

and ρ denotes the distance of the corresponding segments.

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Remark 1. The system of the iterated segments

$$[\varphi_1^n(T),\varphi_1^n(T+U)]\,,\ \left[\varphi_1^{n-1}(T),\varphi_1^{n-1}(T+U)\right],\dots,[T,T+U]$$

is the disconnected set of segments distributed from right to left (see (C)) and the neighbouring segments unboundedly recede each from other (see (D), $\rho\to\infty$, as $T\to\infty$, comp. [6], Remark 3).

Remark 2. Let us mention the formula (1.3) especially for the prime numbers of Fermat-Gauss $n=17,257,65537$ and for the Skewes constant

$$n=10^{10^{10^{34}}}.$$

It is obvious that nor the formula ($n=2$)

$$\int_T^{T+U} Z^2[\varphi_1(\varphi_1(t))]Z^2[\varphi_1(t)]Z^2(t)\mathrm{d}t\sim U\ln^3 T$$

cannot be reached in known theories of Balasubramanian, Heath-Brown and Ivic (see [1]).

This paper is a continuation of the series of papers [2]-[7].

2. Consequences of the Theorem

Using the mean-value theorem in (1.3) we obtain

Corollary 1.

$$\prod_{k=0}^n Z^2[\varphi_1^k(\tau)]\sim\ln^{n+1}T, \tag{2.1}$$

$$T\to\infty, \varphi_1^k(\tau)\in(\varphi_1^k(T),\varphi_1^k(T+U)), k=0,1,\dots,n, \tau=\tau(T,n).$$

From (2.1) we obtain

Corollary 2.

$$\prod_{k=0}^n\left|Z[\varphi_1^k(\tau)]\right|^{\frac{2}{n+1}}\sim\ln T, \tag{2.2}$$

$$\frac{1}{n+1}\sum_{k=0}^n\ln|Z[\varphi_1^k(\tau)]|\sim\frac12\ln\ln T. \tag{2.3}$$

Next, by the known inequalities for harmonic, geometric and arithmetic means we have

Corollary 3.

$$(1-\epsilon)\ln T\le\frac{1}{n+1}\sum_{k=0}^n|Z[\varphi_1^k(\tau)]|^{\frac{2}{n+1}}, \tag{2.4}$$

$$\frac{1}{(1+\epsilon)\ln T}<\frac{1}{n+1}\sum_{k=0}^n|Z[\varphi_1^k(\tau)]|^{-\frac{2}{n+1}}. \tag{2.5}$$

Remark 3. Some new type of the nonlocal interaction of the values

$$\left\{ Z^2[\varphi_1^k(t)] \right\}_{k=0}^n$$

of the signal

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

over the system of disconnected segments

$$\bigcup_{k=0}^n [\varphi_1^k(T), \varphi_1^k(T+U)]$$

is expressed by formulae (1.3), (2.1)-(2.5) for the iterated Jacob's ladder $\varphi_1^k(t)$.

3. Lemma

We start with the formula (see [2], (3.5), (3.9))

$$Z^2(t) = \Phi'_\varphi[\varphi(t)]\frac{\mathrm{d}\varphi(t)}{\mathrm{d}t},\ t \geq T_0[\varphi],$$

where (see [4], (1.5))

$$\Phi'_\varphi[\varphi(t)] = \frac{1}{2}\left\{1 + \mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t.$$

Next we have

$$\tilde{Z}^2(t) = \frac{\mathrm{d}\varphi_1(t)}{\mathrm{d}t},\ t \in [T, T+U],\ U \in \left(0, \frac{T}{\ln T}\right], \tag{3.1}$$

by (1.1) we have

$$\tilde{Z}^2(t) = \frac{Z^2(t)}{2\Phi'_\varphi[\varphi(t)]} = \frac{Z^2(t)}{\left\{1+\mathcal{O}\left(\frac{\ln\ln t}{\ln t}\right)\right\}\ln t}. \tag{3.2}$$

Then we obtain from (3.1) the following lemma (comp. [6], (2.5)).

Lemma. For every integrable function (in the Lebesgue sense) $f(x), x \in [\varphi_1(T), \varphi_1(T+U)]$ the following is true

$$\int_T^{T+U} f[\varphi_1(t)]\tilde{Z}^2(t)\mathrm{d}t = \int_{\varphi_1(T)}^{\varphi_1(T+U)} f(x)\mathrm{d}x,\ U \in \left(0, \frac{T}{\ln T}\right]. \tag{3.3}$$

4. Proof of the Theorem

4.1. From the formula (see (1.1), (1.2), [2], (6.2))

$$t - \varphi_1^1(t) \sim (1-c)\pi(t) \sim (1-c)\frac{t}{\ln t} \tag{4.1}$$

we have

$$\begin{aligned} &\varphi_1^1(t) - \varphi_1^2(t) \sim (1-c)\frac{t}{\ln t}, \\ &\varphi_1^2(t) - \varphi_1^3(t) \sim (1-c)\frac{t}{\ln t}, \\ &\vdots \\ &\varphi_1^n(t) - \varphi_1^{n+1}(t) \sim (1-c)\frac{t}{\ln t}, \end{aligned} \tag{4.2}$$

and by (4.1) (4.2) we obtain

$$\varphi_1^{n+1}(t) > t\left\{1-\frac{(1+\epsilon)(1-c)(n+1)}{\ln t}\right\} \geq T\left(1-\frac{n+1}{\ln T}\right) \geq \frac{1}{2}T \geq T_0[\varphi_1],$$

i.e. (A).

4.2. By comparison of the formula (see [3], (1.5))

$$\int_T^{T+U} Z^2(t)\mathrm{d}t \sim U\ln T,\ U=T^{1/3+2\epsilon},$$

where 1/3 is the exponent of Balasubramanian, and our formula

$$\int_T^{T+U} Z^2(t)\mathrm{d}t \sim \{\varphi_1(T+U)-\varphi_1(T)\}\ln T,$$

(see [4], (1.2)) we have

$$\varphi_1(T+U)-\varphi_1(T) \sim U,$$

by comparison in the cases $T\to\varphi_1^1(T),\ T+U\to\varphi_1^1(T+U),\ \dots$ we obtain

$$\begin{aligned} \varphi_1^2(T+U)-\varphi_1^2(T) &\sim \varphi_1^1(T+U)-\varphi_1^1(T), \\ &\vdots \\ \varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T) &\sim \varphi_1^n(T+U)-\varphi_1^n(T), \end{aligned}$$

i.e. (B).

4.3. By (4.2), $t\to T$ we have

$$\varphi_1^1(T)-\varphi_1^2(T) \sim (1-c)\frac{T}{\ln T},$$

i.e.

$$\varphi_1^1(T)-\varphi_1^2(T+U)+\{\varphi_1^2(T+U)-\varphi_1^2(T)\} \sim (1-c)\frac{T}{\ln T}, \tag{4.3}$$

since (see (B))

$$\varphi_1^2(T+U)-\varphi_1^2(T) \sim U = T^{1/3+2\epsilon}$$

then from (4.3) the asymptotic formula

$$\varphi_1^1(T)-\varphi_1^2(T+U) \sim (1-c)\frac{T}{\ln T}$$

follows. Similarly we obtain all asymptotic formulae in (C). The proposition (D) follows from (C).

4.4. From (3.1) by (3.3) we have

$$\begin{aligned}
&\int_T^{T+U}\prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t=\int_T^{T+U}\prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)]\tilde{Z}^2(t)\mathrm{d}t= \\
&=\int_T^{T+U}\prod_{k=1}^n \tilde{Z}^2[\varphi_1^{k-1}(\varphi_1(t))]\mathrm{d}\varphi_1(t)=\int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)}\prod_{k=1}^n \tilde{Z}^2[\varphi_1^{k-1}(w_1)]\mathrm{d}w_1= \\
&=\int_{\varphi_1^1(T)}^{\varphi_1^1(T+U)}\prod_{k=2}^n \tilde{Z}^2[\varphi_1^{k-2}(\varphi_1(w_1))]\tilde{Z}^2(w_1)\mathrm{d}w_1= \\
&=\int_{\varphi_1^2(T)}^{\varphi_1^2(T+U)}\prod_{k=2}^n \tilde{Z}^2[\varphi_1^{k-2}(w_2)]\mathrm{d}w_2=\dots= \\
&=\int_{\varphi_1^n(T)}^{\varphi_1^n(T+U)}\tilde{Z}^2[w_n]\mathrm{d}w_n=\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T),
\end{aligned}$$

i.e. the following asymptotic formula (see (B))

$$\int_T^{T+U}\prod_{k=0}^n \tilde{Z}^2[\varphi_1^k(t)]\mathrm{d}t=\varphi_1^{n+1}(T+U)-\varphi_1^{n+1}(T)\sim U \tag{4.4}$$

holds true. Then, from (4.4) by mean-value theorem (see (3.1), (3.2), (4.1), (4.2); $\ln\varphi_1^k(t)\sim\ln t$) the formula (1.3) follows.

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